Pergamon

J. Appl. Maths Mechs, Vol. 62, No. 6, pp. 893–900, 1998

0021–8928/98/$—see front matter

PII: S0021–8928(98)00114-2

THE DYNAMICS OF A PARTICLE ON A SMOOTH VIBRATING SURFACE†

V. I. YUDOVICH

Rostov-on-Don

(*Received* 27 *January* 1998)

The van der Pol–Krylov–Bogolubov averaging method [1–5], in a form described earlier [6], for systems with constraints, is used to investigate the dynamics of a particle on a smooth surface which is performing rapid vibration in a conservation force field. The general case of a solid vibrating surface is investigated, and also the case of an ellipsoid with pulsating axes. © 1999 Elsevier Science Ltd. All rights reserved.

It is not always useful, when considering a mechanical system with constraints to change to Lagrange equations of the second kind, eliminating the constraint equations. It is often impracticable. Hence, in many applications, it is better to use the Lagrange equations of the first kind, including the constraint reactions. In this paper we consider ideal holonomic constraints which depend on time $(2\pi/\omega)$-periodically, where the frequency ω is very high. We will assume that the amplitude of the velocity oscillations remains bounded when $\omega \to \infty$.

Under these conditions, the motion can be split into fast and slow components, and the fast component can then be expressed in terms of the slow one, while the evolution of the slow component can be described by averaged equations of motion. The constraint equation is also averaged over fast time. The averaged equations contain a new force, which arises as a result of the non-linear interaction of the vibrations, and is referred to as the vibrogenic force. The term "vibration force", widely used in the literature, seems inappropriate, since this force is autonomous (it is independent of the fast time), although it has a vibration origin.

In this paper we analyse two particular problems. The first of these is the motion of a particle in a gravitational field along an arbitrary smooth surface, which vibrates while keeping its shape. When the surface is a sphere (or a circle on a plane) the classical problem of the motion of a pendulum with a vibrating pivot is obtained ([1], see also [2–5]). Note that the use of Lagrangian curvilinear coordinates, even when this is possible, usually requires the introduction of transcendental functions, which gives rise to well-known difficulties in numerical analysis. The formalism used here is free from this drawback. Its effectiveness is demonstrated using the example of an elliptic pendulum.

The second problem is the motion of a particle in a gravitational field along an ellipsoid, which vibrates about a certain equilibrium position; both rotational vibration and oscillations of the axes of the ellipsoid are allowed.

In both cases, we indicate the limitations on the vibration, for which equilibrium is preserved irrespective of its intensity (as is well known, $(2\pi/\omega)$-periodic modes of the original system correspond to equilibria of the averaged system). Since the vibrogenic force in the averaged equations is a potential force [6], Lagrange's theorem is applicable for their stability. The results obtained indicate diverse possibilities for the vibration equation as regards the stability of equilibrium.

In this paper we consider a system without friction. Note that the Rayleigh friction force, which is linear in the velocity, enters without change into the averaged equations, making the stable equilibria asymptotically stable and preserving the instability of the equilibria.

1. FORMULATION OF THE PROBLEM AND THE AVERAGE EQUATIONS

The motion of a particle in a conservative force field with potential energy V along a vibrating surface will be described by the Lagrange equations of the first kind

$$\ddot{x} = -\nabla V(x) - \Lambda\nabla\Phi \tag{1.1}$$

†*Prikl. Mat. Mekh.* Vol. 62, No. 6, pp. 968–976, 1998.

$$\Phi(x, \tau) = 0, \tau = \omega t \tag{1.2}$$

Here x is an unknown vector function with values $x(t)$ in R^3, Λ is the Lagrange multiplier, and $\Lambda\nabla\Phi$ is the reaction of the ideal constraint (1.2). It is assumed that Φ depends 2π-periodically on the fast time τ. The frequency ω is a large parameter.

We will assume in addition, that the function Φ as $\omega \to \infty$, allows for the asymptotic form

$$\Phi(x,\tau) = \overline{\Phi}(x) + \varepsilon\varphi(x,\tau) + O(\varepsilon^2), \ \ \varepsilon = 1/\omega \tag{1.3}$$

where the mean $\phi(x) = 0$ for all x. Taking into account the 2π-periodicity, the mean is defined by the equation

$$\overline{\Phi}(x) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(x,\tau)d\tau \tag{1.4}$$

Note that when $\bar{\Phi}$ depends smoothly on ε it is sufficient, henceforth, to use the value $\bar{\Phi}$ when $\varepsilon = 0$.

We will further assume that the equation

$$\bar{\Phi}(x) = 0 \tag{1.5}$$

defines a closed smooth surface Γ in R^3 where $\nabla\Phi(x) \neq 0$ when $x \in \Gamma$.

At each point $x \in \Gamma$ we will define projectors P_x and Q_x onto the tangential plane and onto the normal respectively

$$P_x h = h - \zeta^{-2}(h,\zeta)\zeta, \ \ Q_x h = \zeta^{-2}(h,\zeta)\zeta, \ \ \zeta = \nabla\overline{\Phi}, \ \ \zeta^2 = (\zeta,\zeta) \tag{1.6}$$

for any vector field h on Γ.

We will further consider the problem of the $(2\pi/\omega)$-periodic solutions of system (1.1) and (1.2), and also the Cauchy problem

$$x(0) = x_0, \ \dot{x}(0) = \upsilon_0 \tag{1.7}$$

In order that these conditions should be compatible with the constraint, it is necessary for the following equations to be satisfied

$$\Phi(x_0,0) = 0, \ \ \omega\Phi_\tau(x_0,0) + (\nabla\Phi(x_0,0),\upsilon_0) = 0$$

The asymptotic solution of both problems (x_0 and ν_0 are independent of ω) has the form [2, 3]

$$x = \bar{x}(t) + \omega^{-1}\xi(\tau,\bar{x}), \ \ \Lambda = \overline{\Lambda}(t) + \omega\lambda(\tau,\bar{x}) \tag{1.8}$$

The fast unknowns ξ, λ are found from the system

$$\xi'' = -\lambda\zeta, \ \ \bar{\xi} = 0, \ \ \bar{\lambda} = 0 \tag{1.9}$$

$$(\xi,\zeta) + \varphi = 0, \ \ \zeta = \zeta(\bar{x}) = \nabla\overline{\Phi}(\bar{x}) \tag{1.10}$$

The prime denotes differentiation with respect to τ; $\bar{x}$ is present as a parameter. The solution of system (1.9), (1.10) has the form

$$\xi = -\mu\zeta, \ \ \mu = \varphi\zeta^{-2}, \ \ \lambda = \varphi''\zeta^{-2} \tag{1.11}$$

where μ is a function, which is uniquely defined by the conditions

$$\mu'' = \lambda, \ \ \bar{\mu} = 0 \tag{1.12}$$

Substituting (1.8) into (1.1) and (1.2) and taking (1.11) into account we obtain the averaged system [6]

$$\ddot{\bar{x}} = -\nabla V(\bar{x}) - \nabla V_\varphi(\bar{x}) - \overline{\Lambda}\nabla\overline{\Phi} \tag{1.13}$$

$$\overline{\Phi}(\bar{x}) = 0 \tag{1.14}$$

The second term in (1.13) is the vibrogenic force, and the vibrogenic potential energy corresponding to it is

$$V_\varphi = -1/2\overline{\xi'^2} = -1/2\overline{\varphi'^2}\zeta^{-2} \tag{1.15}$$

If we solve the Cauchy problem, the asymptotic initial conditions must be taken in the form

$$\bar{x}(0) = x_0, \quad \dot{\bar{x}}(0) = v_0 - \xi'(0, x_0) = v_0 + \varphi'(0, x_0)\zeta^{-2}\zeta$$

Here the condition of tangency of the average velocity $\dot{\bar{x}}(0)$ to the surface (1.14) is satisfied by virtue of (1.10); the equilibrium of the averaged system (1.13), (1.14) corresponds to the $(2\pi/\omega)$-periodic solutions of the initial system.

The remaining part of this paper is devoted to analysing the stability of the equilibria of system (1.13), (1.14) when a gravitational force acts on the particle such that

$$V = V_g = (-\mathbf{g}, x) \tag{1.16}$$

The vertical axis is directed upwards. The acceleration vector of the gravitational force is then $\mathbf{g} = (0, 0, -g)$.

2. A PARTICLE ON A SOLID VIBRATING SURFACE

We will assume that a point mass moves along a surface

$$\Phi(W_\varepsilon(\tau)x) = 0 \tag{2.1}$$

where $W_\varepsilon(\tau)$ is a smooth family of motions (isometric transformations) of the space R^3, defined for all $\tau \in R$ and for small $\varepsilon \in R$. The dependence on τ is assumed to be 2π-periodic. We will assume that for small ε the following asymptotic equation is satisfied uniformly with respect to $\tau \in R$

$$W_\varepsilon(\tau) = I + \varepsilon U(\tau) + O(\varepsilon^2), \quad \varepsilon \to 0 \tag{2.2}$$

where $\overline{U} = 0$. Hence, the surface (2.1), without changing its shape, executes rapid vibration about a certain mean position.

The equation of motion (1.1) for a particle acted upon by a gravitational force has the form

$$\ddot{x} = \mathbf{g} - \Lambda\nabla\Phi \tag{2.3}$$

If we ascribe to the parameter ε the meaning of time, from the equation

$$U(\tau)x = d/d\varepsilon\,|_{\varepsilon=0}\, W_\varepsilon(\tau)x \tag{2.4}$$

it will be seen that $U(\tau)$, like the velocity of a solid, allows of the representation

$$U(\tau)x = \eta(\tau) + S(\tau)x \tag{2.5}$$

Here $\eta(\tau)$ is independent of x and specifies translational motion, while the operator function $S(\tau)$ corresponds to rotational motion. For each τ the operator $S(\tau)$ is skew symmetric: $S^*(\tau) = -S(\tau)$. In the case of R^3 we can introduce the pseudovector $\mathbf{q}(\tau)$ of the angular velocity, in which case $S(\tau)x = \mathbf{q} \times x$.

We will now specify in more detail the vibrogenic potential energy (1.15) for the case considered. The constraint equation (2.1), in view of (2.2), can be written as follows:

$$\Phi(x) + \varepsilon(U(\tau)x, \zeta(x)) + O(\varepsilon^2) = 0, \quad \varepsilon \to 0; \quad \zeta(x) = \nabla\Phi(x) \tag{2.6}$$

i.e. in this case

$$\varphi(x, \tau) = (\zeta, U(\tau)x) = (\zeta, \eta(\tau)) + (\zeta, S(\tau)x) \tag{2.7}$$

Substituting (2.7) into (1.15) we obtain the representation of V_φ in the form

$$V_\varphi = V_\varphi^\eta + V_\varphi^S + V_\varphi^{\eta S} \tag{2.8}$$

$$V_\varphi^\eta = -\tfrac{1}{2}\overline{(n(x), \eta')^2}, \quad V_\varphi^S = -\tfrac{1}{2}\overline{(n(x), S'x)^2}, \quad V_\varphi^{\eta S} = -\overline{(n, \eta')(n, S'x)}$$

Here $n = n(x) = \zeta(x)/|\zeta|(\zeta = \nabla\Phi)$ is the field of the unit vectors normal to Γ, defined in a certain neighbourhood of the surface Γ: $\Phi(x) = 0$. The terms V_φ^η, V_φ^S, $V_\varphi^{\eta S}$ denote the contributions to the vibrogenic potential energy from the translational vibration, the rotational vibration and from their interaction, respectively.

The averaged equations are written in the form

$$\ddot{\bar{x}} = \mathbf{g} - \nabla V_\varphi(\bar{x}) - \bar{\Lambda}\nabla\Phi(\bar{x}), \quad \Phi(\bar{x}) = 0 \tag{2.9}$$

In coordinate form we have the expressions

$$\begin{aligned} V_\varphi^\eta &= -\tfrac{1}{2} n_i(x) n_k(x)\varepsilon_{ik}, \quad \varepsilon_{ik} = \overline{\eta_i'\eta_k'} \\ V_\varphi^S &= -\tfrac{1}{2}\varkappa_{ik}(x\times n)_i(x\times n)_k, \quad \varkappa_{ik} = \overline{q_i' q_k'} \\ V_\varphi^{\eta S} &= -\nu_{ik} n_i (x\times n)_k, \quad \nu_{ik} = \overline{\eta_i' q_k'} \end{aligned} \tag{2.10}$$

Here $\eta = (\eta_1, \eta_2, \eta_3)$ defines the translational velocity of the vibration η^1, while $\mathbf{q} = (q_1, q_2, q_3)$ defines the angular velocity $\mathbf{q}'$. The matrices (ε_{ik}) and $(\varkappa_{ik})$ are symmetrical and positive (although not always positive definite). It emerges that the general vibration is specified by $6 + 6 + 9 = 21$ parameters ε_{ik}, $\varkappa_{ik}$ $(1 \leqslant k \leqslant 3)$ and ν_{ik} $(1 \leqslant i \leqslant k \leqslant 3)$. It can be proved that they are independent, so that any of the combinations can be realised by an appropriate choice of the vibration.

Stabilizing and destabilizing action of the vibration. We will consider the case when the equilibrium $x_0 = 0$ (which is taken as the origin of coordinates), which occurs when there is no vibration, is also preserved when it is present, and, moreover, regardless of the absolute values of the vector η and the pseudo-vector $\mathbf{q}$. We will obtain the conditions for such conservation and we will investigate how the vibration affects the stability.

Thus, we will assume that $\Phi(0) = 0$, and, at the point $x = 0$, the equilibrium equation

$$\mathbf{g} - \Lambda\nabla\Phi = 0, \quad \Lambda = (\mathbf{g}, n)/|\nabla\Phi| \tag{2.11}$$

is satisfied.

The equilibrium $x_0 = 0$ is preserved after introducing the vibration if the vibrogenic force ∇V_φ at the point $x = 0$ is vertical.

In the neighbourhood of zero, the surface Γ can be specified by the equation $x_3 = F(x_1, x_2)$. By rotating the x_1 and x_2 axes in the horizontal plane we can reduce the second differential of the function F to the sum of squares and write the equation of the surface in the form

$$x_3 = F(x_1, x_2) = \frac{1}{2}(b_1 x_1^2 + b_2 x_2^2) + \frac{1}{6}(c_{30}x_1^3 + c_{21}x_1^2 x_2 + c_{12}x_1 x_2^2 + c_{03}x_2^3) + \ldots \tag{2.12}$$

Correspondingly, for the components of the normal field

$$n = (n_1, n_2, n_3) = \frac{(-F_{x_1}, -F_{x_2}, 1)}{(1 + F_{x_1}^2 + F_{x_2}^2)^{1/2}}$$

we obtain the following expansions in Taylor series up to terms of the second order

$$\begin{aligned} n_1 &= -b_1 x_1 + n_1^{(2)}(x_1, x_2) + \ldots; \quad n_1^{(2)} = -\tfrac{1}{6}(3c_{30}x_1^2 + 2c_{21}x_1 x_2 + c_{12}x_2^2) \\ n_2 &= -b_2 x_2 + n_2^{(2)}(x_1, x_2) + \ldots; \quad n_2^{(2)} = -\tfrac{1}{6}(c_{21}x_1^2 + 2c_{12}x_1 x_2 + 3c_{03}x_2^2) \\ n_3 &= 1 + n_3^{(2)}(x_1, x_2) + \ldots; \quad n_3^{(2)} = -\tfrac{1}{2}(b_1^2 x_1^2 + b_2^2 x_2^2) \end{aligned}$$

Substituting these expansions into (2.10) we obtain

$$\begin{aligned} 2V_\varphi^\eta &= -\varepsilon_{33} + 2(\varepsilon_{13}b_1 x_1 + \varepsilon_{23}b_2 x_2) + (\varepsilon_{33} - \varepsilon_{11})b_1^2 x_1^2 - \\ &\quad -2b_1 b_2 \varepsilon_{12} x_1 x_2 + (\varepsilon_{33} - \varepsilon_{22})b_2^2 x_2^2 - 2(\varepsilon_{13}n_1^{(2)} + \varepsilon_{23}n_2^{(2)}) + \ldots \\ 2V_\varphi^S &= -\varkappa_{22}x_1^2 + 2\varkappa_{12}x_1 x_2 - \varkappa_{11}x_2^2 + \ldots \end{aligned} \tag{2.13}$$

$$V_\varphi^{\eta S} = \nu_{32}x_1 - \nu_{31}x_2 - \nu_{12}b_1x_1^2 + [(\nu_{11} - \nu_{33})b_1 + (\nu_{33} - \nu_{22})b_2]x_1x_2 + \nu_{21}b_2x_2^2 + \dots$$

In order that the zeroth equilibrium of the averaged equations (2.9) should be preserved when the vibration is included, the following conditions must be satisfied

$$\frac{\partial}{\partial x_i}V_\varphi(x_1, x_2, F(x_1, x_2)) = 0, \quad i = 1, 2$$

when $x_1 = x_2 = 0$. By (2.13), these conditions take the form

$$\varepsilon_{13}b_1 + \nu_{32} = 0, \quad \varepsilon_{23}b_2 - \nu_{31} = 0 \tag{2.14}$$

In this case we have the following expansion for the vibrogenic potential energy

$$\begin{aligned} &2V_\varphi = -\varepsilon_{33} + px_1^2 + 2qx_1x_2 + rx_2^2 + \dots \\ &p = (\varepsilon_{33} - \varepsilon_{11})b_1^2 - \varkappa_{22} - 2\nu_{12}b_1 + \varepsilon_{13}c_{30} + \tfrac{1}{3}\varepsilon_{23}c_{21} \\ &q = \varkappa_{12} - b_1b_2\varepsilon_{12} + b_1(\nu_{11} - \nu_{33}) + b_2(\nu_{33} - \nu_{22}) + \tfrac{1}{3}(\varepsilon_{13}c_{21} + \varepsilon_{23}c_{12}) \\ &r = (\varepsilon_{33} - \varepsilon_{22})b_2^2 - \varkappa_{11} + 2\nu_{21}b_2 + \tfrac{1}{3}\varepsilon_{13}c_{12} + \varepsilon_{23}c_{03} \end{aligned} \tag{2.15}$$

Since $V_g = gx_3 = gF(x_1, x_2)$, from (2.12) and (2.13) we obtain the following expansion for the total potential energy $V = V_g = V_\varphi$

$$2V = -\varepsilon_{33} + (p + gb_1)x_1^2 + 2qx_1x_2 + (r + gb_2)x_2^2 + \dots$$

By Lagrange's theorem, the equilibrium $x_0 = 0$ is stable if it is a point of strict minimum for V. For this to be the case it is sufficient for the following inequalities to be satisfied

$$p + gb_1 > 0, \quad r + gb_2 > 0, \quad (p + gb_1)(r + gb_2) - q^2 > 0 \tag{2.16}$$

On the other hand, it follows from Lyapunov's theorem on the inversion of Lagrange's theorem [7] that the equilibrium is unstable if at least one of these inequalities is replaced by the opposite strict inequality. Note that the second inequality of (2.16) follows from the remaining two.

We will give the simplest example of the use of these results Suppose a particle (a small solid bead) is constrained to move along an ellipse (a thin closed tube), which, without changing its shape and dimensions, executes vibrational motion in the x_1, x_3 plane. Then in (2.5) $\eta = (\eta_1, 0, \eta_3)$, $\mathbf{q} = (0, q_2, 0)$. We will assume that in the middle position one of the axes of the ellipse is vertical, so that its equation has the form

$$\Phi(x) \equiv x_1^2/a_1^2 + x_3^2/a_3^2 - 1 = 0 \tag{2.17}$$

In order to use the criteria of stability (2.16), we must transfer the origin of coordinates to the position of equilibrium x^0 being investigated. This is achieved by making the replacement $x \to x + x^0$ and correspondingly $\eta \to \eta + \mathbf{q} \times x^0$, $\mathbf{q} \to \mathbf{q}$ in expressions (2.10) for all ε_{ik}, $\varkappa_{ik}$, ν_{ik}. Of these, only ε_{11}, ε_{13}, ε_{33}, $\varkappa_{22}$, ν_{12}, ν_{32}, can be non-zero in this case.

The upper position of equilibrium $x^0 = (0, a_3)$, by (2.14), is preserved during vibration, if the following equation is satisfied

$$\delta\varepsilon_{13} - a_1(1 - \delta^2)\nu_{32} = 0, \quad \delta = a_3/a_1$$

Here we have taken into account the fact that the local equation of the ellipse (2.7) in the region of the point (0, a_3) has the form $x_3 = a_3 - x_1^2a_3/(2a_1^2) + \dots$ so that $b_1 = -\delta/a_1$. The condition of stability is the first inequality of (2.16), which we can now write in the form

$$\delta^2(\varepsilon_{33} - \varepsilon_{11}) - (1 - \delta^2)^2a_1^2\varkappa_{22} + 2a_3(1 - \delta^2)\nu_{12} - ga_3 > 0 \tag{2.18}$$

It can also be written in the form

$$\delta^2\varepsilon_{33} - \overline{[\delta\eta_1' - (1 - \delta^2)a_1q_2']^2} - ga_3 > 0 \tag{2.19}$$

Hence it follows that by increasing the intensity of the vertical vibration we can achieve stability of the upper position of equilibrium. As regards the horizontal and rotational vibrations, their influence always destabilizes the upper equilibrium, but it reduces to nothing when $\delta\eta'_1(\tau) = (1 - \delta^2)q_2^1 a_1$ for all τ.

The results for the lower equilibrium are obtained similarly. The condition for it to be preserved is $\delta\varepsilon_{13} + a_1(1 - \delta^2)\nu_{32} = 0$, so that both equilibria are preserved when $\varepsilon_{13} = 0$ and $\nu_{32} = 0$. It is stable when a condition, differing from (2.19) by the replacement of a_1 and a_3 by $-a_1$ and $-a_3$, is satisfied.

As can be seen, the vertical vibration makes it even more stable, but it may be destabilized by the action of the horizontal and rotational vibrations.

When $a_1 = a_3$ we obtain the classical results [1–5] for a circular pendulum.

It follows from (2.15) and (2.16) that the vertical vibration of the pendulum stabilizes while the horizontal and rotational vibrations destabilize. At the same time, their interaction may have both a stabilizing and destabilizing effect.

3. A POINT MASS ON A PULASTING ELLIPSOID

We will specify an ellipsoid in R^m, deforming with time, by the equation

$$\Phi(x, \tau) = \tfrac{1}{2}[(A_\varepsilon(\tau)x, x) - 1] = 0 \tag{3.1}$$

where A_ε for all τ and small $\varepsilon = 1/\omega$ is a positive-definite linear operator. We will assume that it depends 2π-periodically on the fast time $\tau = \omega t$, and, as $\varepsilon \to 0$, it allows of the asymptotic representation

$$A_\varepsilon(\tau) = A_0 + \varepsilon S(\tau) + O(\varepsilon^2)$$

Here $\bar{S} = 0$ and the operator A_0 are positive-definite. Correspondingly, we can write Eq. (3.1) in the form

$$\Phi_0(x) + \varepsilon\varphi(x, \tau) = O(\varepsilon^2)$$

$$\Phi_0(x) = \tfrac{1}{2}[(A_0 x, x) - 1], \quad \varphi(x, \tau) = \tfrac{1}{2}(S(\tau)x, x)$$

The vibrogenic potential energy of the particle, which moves along the ellipsoid (3.1), can be calculated from (1.15). Taking into account the equalities $\zeta(x) = \nabla\Phi_0(x) = A_0 x$ we obtain (in tensor notation)

$$V_\varphi = -\frac{(Wx \otimes x, x \otimes x)}{8(A_0 x, A_0 x)}, \quad W = \overline{S'(\tau) \otimes S'(\tau)} \tag{3.2}$$

Hence, V_φ is defined by the operator W, which acts in the tensor square $R^m \otimes R^m = R^{m2}$. Its matrix is the averaged Kronecker square of the matrix $(s'_{ik}(\tau))$ of the operator $S'(\tau)$. In coordinates we have

$$V_\varphi = -\sum_{i,j,k,l=1}^{m} \overline{s'_{ij}s'_{kl}} x_i x_j x_k x_l \left(8 \sum_{i,j,k=1}^{m} a_{ij} a_{jk} x_i x_k\right)^{-1} \tag{3.3}$$

where (a_{ij}) is the matrix of the operator A_0. Since the operator $S(\tau)$ is symmetrical for all τ, among the coefficients in the numerator there are in all $q(a + 1)/2$ $(q = m(m + 1)/2)$ different ones. By choosing the vibration appropriately we can obtain an arbitrary set of these parameters.

We will consider in more detail the case when all the semiaxes of the ellipsoid retain their direction and one of them is vertical. The following equations are then satisfied

$$A_0 e_j = a_j^{-2} e_j, \quad S(\tau)e_j = s_j(\tau)e_j, \quad j = 1, \ldots, m$$

where $e_1, \ldots, e_m$ are vectors of a canonical basis in R^m, and we have denoted the lengths of the semiaxes of the ellipsoid E by $a_1, \ldots, a_m$: $(A_0 x, x) = 1$, while $s_1(\tau), \ldots, s_m(\tau)$ are the eigenvalues of the operator $S(\tau)$. Expression (3.3) is simplified and becomes

$$V_\varphi = -\sum_{i,j=1}^{m} \gamma_{ij} x_i^2 x_j^2 \left(8 \sum_{j=1}^{m} \frac{x_j^2}{a_j^4}\right)^{-1}, \quad \gamma_{ij} = \overline{s'_i s'_j} \tag{3.4}$$

If the j-th semiaxis of the ellipsoid oscillates as $a_j + \varepsilon\eta_j(\tau) + O(\varepsilon^2)$, we have $s_j = -2_j/a_j^3$.

The averaged equations (1.13) and (1.14) can be written in the form

$$\ddot{x} = ge_m - \nabla V_\varphi - \bar{\Lambda}\nabla\Phi_0, \quad \Phi_0(\bar{x}) = 0$$

When there is no vibration there are exactly two equilibria: an upper one $x^u = (0, \ldots, 0, a_m)$ and a lower one $x^l = (0, \ldots, 0, -a_m)$. To preserve these it is necessary for the vibrogenic force in them to be vertical. This condition is obviously satisfied in the case of the potential energy (3.4).

We will now investigate the effect of vibration on the stability of the equilibria x^l and x^u. By Lagrange's theorem, for the equilibrium to be stable it is sufficient for the potential energy V to reach a strict minimum on it. As Lyapunov showed, when the second differential d^2V is non-degenerate, this condition is also the necessary condition (see [7], where some degenerate cases are also considered).

Confining ourselves to the case when the equilibria are non-degenerate, we will calculate the second differential d^2V of the total potential energy $V = V_\varphi + V_g$ at the points x^l and x^u on the ellipsoid E. In the neighbourhood of each of these points, we can take $x_1, \ldots, x_{m-1}$ as the coordinates on E. Expressing x_m in terms of $x_1, \ldots, x_{m-1}$ in (3.4) from the equation of the ellipsoid E: $(A_0x, x) = 1$ and expanding V_φ in a Taylor series up to second-order terms, we have for both equilibria

$$V_\varphi = -\frac{1}{8}\gamma_{mm}a_m^6 + \sum_{i=1}^{m-1}\varkappa_i^\varphi x_i^2 + \ldots$$

$$\varkappa_i^\varphi = \frac{1}{8}a_m^4[\gamma_{mm}(\varepsilon_i^2 + \varepsilon_i^4) - \gamma_{im}], \quad \varepsilon_i = \frac{a_m}{a_i}$$

For the gravitational potential energy V_g we similarly obtain

$$V_g = \pm ga_m + \sum_{i=1}^{m-1}\varkappa_i^g x_i^2 + \ldots, \quad \varkappa_i^g = \mp\frac{ga_m}{2a_i^2}$$

The upper signs correspond to the upper equilibrium while the lower signs correspond to the lower equilibrium.

The equilibrium is Lyapunov-stable if all its Poincaré stability coefficients (see [7]) $\varkappa_i = \varkappa_i^\varphi + \varkappa_i^g$ are strictly positive, and unstable if at least one of these is negative. As a result we obtain that the equilibrium is stable if the following strict inequalities are satisfied

$$\gamma_{mm}(\varepsilon_i^2 + \varepsilon_i^4) - 2\gamma_{im} \mp 4g/(a_i^2a_m^3) > 0, \quad i = 1, \ldots, m-1 \tag{3.5}$$

Here the upper sign relates to x^u and the lower sign relates to x^l. The equilibrium is unstable if at least one of these inequalities roughly breaks down, so that the > sign is replaced by <.

We will present some conclusions which follow from these conditions.

The pulsation of the vertical semiaxis of the ellipsoid has a stabilizing effect on both equilibria x^l and x^u while the effect of purely horizontal vibration is negligibly small at high frequencies ω. In any case, it only has an effect in the critical case when equality occurs for certain values of i in (3.5). Nevertheless, the interaction of the horizontal and vertical vibrations, determined by the coefficient γ_{im}, may both stabilize the equilibrium (when $\gamma_{im} < 0$), and destabilize the equilibrium (if $\gamma_{im} > 0$). By increasing all the absolute values of $-\gamma_{im} > 0$ $(i = 1, \ldots, m-1)$ one can make both equilibria x^l and x^u stable.

The apparently paradoxical conclusion follows from (3.5) that the stabilizing intensity of the vertical vibration falls when the vertical axis of the ellipsoid increases. This ceases to be strange if we note that for an ellipsoid that is oblate along the vertical the particle has considerable freedom in horizontal motions. In the limiting case of a plane boundary its vertical vibration is, in general, unlimited.

If the upper equilibrium is stable, the lower equilibrium will also be stable. If the lower equilibrium is unstable the upper equilibrium will also be unstable.

When $\gamma_{mm} = 0$ (then also all $\gamma_{im} = 0$) the upper equilibrium is unstable while the lower one is stable. We will increase γ_{mm}, assuming the remaining parameters to be fixed. Then the lower position of equilibrium remains stable, while for large γ_{mm}, as can be seen from (3.5), the upper equilibrium also becomes stable.

Suppose that, for certain values of the parameters γ_{mm} and γ_{im}, both equilibria are unstable. The critical value of the parameter γ_{mm} for this equilibrium is that for which at least one of the stability coefficients vanishes. If γ_{mm}^* is the greatest critical value for x^u, then, when $\gamma_{mm} > \gamma_{mm}^*$ both equilibria are stable. When γ_{mm}, on increasing, intersects the value γ_{mm}^*, then, by Poincaré's theory [7], a pair of stable equilibria of the averaged system—quasi-equilibria of the initial system (or several pairs; in the case of ellipsoids of revolution even a continuum) will branch out from x^u. When γ_{mm} is increased further they may not all disappear—the potential energy V should reach a maximum somewhere while the equilibria x^1 and x^u are local minima for large γ_{mm}.

A similar bifurcation occurs when γ_{mm} increases around the lower equilibrium, if it is unstable for a certain value of γ_{mm}. Obviously the unstable equilibria also bifurcate—each time the sign of one of the stability coefficients changes. When γ_{mm} becomes very large, we have the following asymptotic form for V

$$V = -\frac{1}{8}\gamma_{mm}x_m^4\left(\sum_{i=1}^{m}\frac{x_i^2}{a_i^4}\right)^{-1} + O(1), \quad \gamma_{mm} \to \infty$$

Hence we can conclude that all the equilibria, distinct from x^l and x^u, approach an equatorial ellipsoid $x_m = 0, x_1^2/a_1^2 + x^2 2_/a_2^2 + \ldots + x_{m-1}^2/a_{m-1}^2 = 1$ as $\gamma_{mm} \to \infty$.

As can be seen, in this case also vibration may produce new equilibria of the averaged system.

In the case of weightlessness ($g = 0$) the dynamics of a particle on a pulsating ellipsoid remains fairly high-grade. Note that the vibration of one single axis of the ellipsoid gives stable equilibria at its ends.

I wish to thank I. I. Blekhman, S. M. Zen'kovskii and L. G. Kurakin for useful discussions, and Ye. V. Shirayayev for help with the manuscript.

This research was supported by the Russian Foundation for Basic Research (96-01-07191, 96-15-96081).

REFERENCES

1. KAPITSA, P. L., The dynamic stability of a pendulum with an oscillating point of suspension. *Zh. Eksp. Teor. Fiz.*, 1951, **21**, 5, 588–597.
2. BOGOLYUBOV, N. N. and MITROPOL'SKII, Yu. A., *Asymptotic Methods in the Theory of Non-linear Oscillations*. Nauka, Moscow, 1974.
3. STRIZHAK, T. G., *The Method of Averaging in Problems of Mechanics*. Vishcha Shkola, Kiev, Donetsk, 1982.
4. SANDERS, J. A. and VERHULST, F., *Averaging Methods in Nonlinear Dynamical Systems*. Springer, New York, 1985.
5. BLEKHAM, I. I., *Vibration Mechanics*. Nauka; Izd. Fiz.-Mat. Lit., Moscow, 1994.
6. YUDOVICH, V. I., Vibrational dynamics of systems with constraints. *Dokl. Ross. Akad. Nauk*, 1977, **354**, 5, 622–624.
7. CHETAYEV, N. G., *The Stability of Motion*. Gostekhizdat, Moscow, 1955.

Translated by R.C.G.